THE SMALL PARAMETER METHOD FOR DETERMINING THE MOTION OF THE VISCOUS INCOMPRESSIBLE FLUID IN A SUPPORT BEARING

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The steady-state flow of a viscous incompressible fluid between two cylinders is considered. One of the cylinders is circular and rotates about its axis at the constant angular velocity ω , while the other remains stationary. The latter cylinder is close in shape to some circular cylinder coaxial with the first.

1. Let us introduce the cylindrical coordinate system (r, ϑ, z_1), directing the z_1-axis along the axis of the rotating cylinder. We write the equations of motion of the fluid between the cylinders in dimensionless form, taking as our length scale the characteristic dimension of the outer cylinder and as our time scale the quantity $1/\omega$. According to [1], the equations can then be written as

$$R(\nabla\times\mathbf{V})\times\mathbf{V}=-\nabla H-\nabla\times(\nabla\times\mathbf{V}),\qquad \nabla\mathbf{V}=0,\ H=p+R\mathbf{V}\mathbf{V}/2 \qquad (1.1)$$

Here R is the Reynolds number, $\mathbf{V}$ is a dimensionless velocity vector, ∇ is a Hamiltonian, and p is the hydrodynamic pressure.

The stationary cylinder in the chosen coordinate system is given by Equation $r = a + \varepsilon a\Phi(\upsilon)$, where ε and a are constants. Furthermore, if the outer cylinder is stationary, then $a = 1$; if the inner cylinder is stationary, then $a = r_1 (r_1 < 1)$.

The boundary conditions for Equation (1.1) are as follows:

if it is the inner cylinder which rotates, then

$$\mathbf{V}=\mathbf{E}\quad \text{for}\ r=r_1,\qquad \mathbf{V}=0\quad \text{for}\ r=1+\varepsilon\Phi(\vartheta) \qquad (1.2)$$

if it is the outer cylinder which rotates, then

$$V=0\quad \text{for}\ r=r_1+\varepsilon r_1\Phi(\vartheta),\qquad \mathbf{V}=\mathbf{E}\quad \text{for}\ r=1 \qquad (1.3)$$

Here $\mathbf{E}$ is a vector with the components $E_r = 0,\ E_\vartheta = 1$.

In addition to (1.2) and (1.3), we shall also make use of the periodicity of the hydrodynamic pressure with respect to ϑ.

For small ε and $|\Phi(\vartheta)| <$ const the space occupied by the fluid is not much different from a circular ring. We shall assume that all of the conditions set forth in [2] apply to the curves bounding this space. Let us map conformally the space occupied by the fluid onto a circular ring. The mapping function can be written as $\zeta = \rho e^{i\varphi} = z + \varepsilon\Phi_1(z, \varepsilon)$, where $z = re^{i\vartheta}$. The radius of the inner circle in conformal mapping is fixed at $\rho = 1$, so that the radius $\rho = \beta$ of the outer circle is determined by the data of the problem.

Let us rewrite Equation (1.1) and boundary conditions (1.2) and (1.3) in the new orthogonal coordinate system (ρ, φ), introducing the new functions $\mathbf{V}'$ and p' to be determined by Formulas

$$\mathbf{V}' = J^{1/2}\mathbf{V} - \mathbf{V}_0, \quad p' = p - p_0 \quad \left(J = \left|\frac{d\zeta}{dz}\right|^2 = 1 + \varepsilon\Phi_2(\rho, \varphi)\right)$$

where J is the coordinate transforming Jacobian.

The components of the vector-function V_0 and the function p_0 are given by Expressions

$$u_0 = 0, \quad v_0 = \frac{1}{1-\beta^2}\left(\rho - \frac{\beta^2}{\rho}\right), \quad p_0 = \frac{1}{(1-\beta^2)^2}\left(0.5\rho^2 - 2\beta^2\ln\rho - \frac{0.25\beta^4}{\rho^4}\right) + \text{const} \quad (1.4)$$

if the inner cylinder rotates, and

$$u_0 = 0, \quad v_0 = \frac{\beta}{\beta^2-1}\left(\rho - \frac{1}{\rho}\right), \quad p_0 = \frac{\beta^2}{(\beta^2-1)}\left(0.5\rho^2 - 2\ln\rho - \frac{0.25}{\rho^4}\right) + \text{const} \quad (1.5)$$

if the outer cylinder rotates.

Henceforth we shall assume that $\Phi_2(\rho, \varphi) \in C^{(1)}(\Omega)$, where $\Omega(\rho, \varphi)$ is a ring defined by the inequalities $1 \leqslant \rho \leqslant \beta$, $0 \leqslant \varphi < 2\pi$.

After the indicated substitutions, Equation (1.1) and boundary conditions (1.2) of (1.3) can be rewritten as follows (the prime denoting the unknowns will henceforth be omitted):

$$R[(\nabla\times\mathbf{V}_0)\times\mathbf{V} + (\nabla\times\mathbf{V})\times\mathbf{V}_0 + (\nabla\times\mathbf{V})\times\mathbf{V}] = -\nabla H_1 - \nabla\times(\nabla\times\mathbf{V}) + \varepsilon\mathbf{F}(\mathbf{V}) \quad (1.6)$$

$$\nabla\mathbf{V} = 0, \quad H_1 = p + 0.5R(\mathbf{V}_0\mathbf{V} + \mathbf{V}\mathbf{V}), \quad \mathbf{V} = 0 \quad \text{for } \rho = 1, \quad \rho = \beta$$

$$\mathbf{F}(\mathbf{V}) = -R\Phi_3(\nabla\times\mathbf{V}_0)\times\mathbf{V}_0 - \Phi_3\nabla\times(\nabla\times\mathbf{V}_0) - \nabla\Phi_3\times(\nabla\times\mathbf{V}_0) - R\Phi_3[(\nabla\times\mathbf{V}_0)\times$$
$$\times\mathbf{V} + (\nabla\times\mathbf{V})\times\mathbf{V}_0] - \Phi_3\nabla\times(\nabla\times\mathbf{V}) - R\Phi_3(\nabla\times\mathbf{V})\times\mathbf{V} -$$
$$-\nabla\Phi_3\times(\nabla\times\mathbf{V}), \quad \Phi_3 = -\Phi_2/(1+\varepsilon\Phi_2)$$

We shall attempt to solve problem (1.6) in the form of a series in the small parameter ϵ.

$$\mathbf{V} = \sum_{n=1}^{\infty}\mathbf{V}_n\varepsilon^n, \qquad H_1 = \sum_{n=1}^{\infty}H_{1,n}\varepsilon^n \quad (1.7)$$

Let us substitute the expressions from (1.7) into (1.6) and collect the terms containing the same powers of the parameter ϵ. This yields the following sequence of linear equations and accompanying boundary conditions:

$$R[(\nabla\times\mathbf{V}_0)\times\mathbf{V}_n + (\nabla\times\mathbf{V}_n)\times\mathbf{V}_0] + \nabla H_{1,n} + \nabla\times(\nabla\times\mathbf{V}_n) = \mathbf{f}_n \quad (1.8)$$

$$\nabla\mathbf{V}_n = 0, \quad \mathbf{V}_n = 0 \quad \text{for } \rho = 1, \quad \rho = \beta$$

$$\mathbf{f}_n = -\frac{R}{1+\varepsilon\Phi_2}\sum_{k=1}^{n-1}(\nabla\times\mathbf{V}_k)\times\mathbf{V}_{n-k} - R\Phi_3[(\nabla\times\mathbf{V}_0)\times\mathbf{V}_{n-1} +$$
$$+(\nabla\times\mathbf{V}_{n-1})\times V_0] - \Phi_3\nabla\times(\nabla\times\mathbf{V}_{n-1}) - \nabla\Phi_3\times(\nabla\times\mathbf{V}_{n-1}) + \boldsymbol{\chi}_n \quad (1.9)$$

Here

$$\boldsymbol{\chi}_n = 0 \quad \text{for } n > 1$$

$$\boldsymbol{\chi}_1 = -R\Phi_3(\nabla\times\mathbf{V}_0)\times\mathbf{V}_0 - \Phi_3\nabla\times(\nabla\times\mathbf{V}_0) - \nabla\Phi_3\times(\nabla\times\mathbf{V}_0)$$

2. Let us prove the solvability of problem (1.8). We shall show first that it has unique solution in the space of the vector-function $\mathbf{V}_n \in W_2^2(\Omega)$ and $H_{1,n} \in W_2^1(\Omega)$. It will be assumed from now on that all of the operations in (1.8) are written in a polar coordinate system.

Let us consider problem (1.8) for $\mathbf{f}_n \equiv 0$. We take the scalar product of the first vector equation of this problem and $\partial\mathbf{V}_n/\partial\varphi$ and integrate over the domain Ω . The vector-function $\partial \mathbf{V}_n / \partial\varphi \in W_2^1$ and is solenoidal. By virtue of the orthogonality of solenoidal functions (which vanish on the boundary of the domain) to functions of the form $\nabla H_{1,n}$, we arrive at the following equation:

$$R \int\limits_{\Omega} \frac{\partial \mathbf{V}_n}{\partial\varphi} [(\nabla \times \mathbf{V}_n) \times \mathbf{V}_0 + (\nabla \times \mathbf{V}_0) \times \mathbf{V}_n] \rho d\rho d\varphi + \int\limits_{\Omega} \frac{\partial \mathbf{V}_n}{\partial\varphi} \nabla \times (\nabla \times \mathbf{V}_n) \rho d\rho d\varphi = 0 \quad (2.1)$$

It can be shown that the second integral in this equation can be transformed in the following way with the aid of vector analysis formulas:

$$\int\limits_{\Omega} \frac{\partial \mathbf{V}_n}{\partial\varphi} \nabla \times (\nabla \times \mathbf{V}_n) \rho d\rho d\varphi = \int\limits_{\Omega} \left(\nabla \times \frac{\partial \mathbf{V}_n}{\partial\varphi}\right) (\nabla \times \mathbf{V}_n) \rho d\rho d\varphi + \int\limits_{\Gamma} \left[\frac{\partial \mathbf{V}_n}{\partial\varphi} \times (\nabla \times \mathbf{V}_n)\right]_{\mathbf{n}} d\Gamma$$

Here Γ is the boundary of the domain Ω , and $\mathbf{n}$ is the vector of the normal to Γ . By virtue of the fact that $\partial\mathbf{V}_n/\partial\varphi = 0$ on Γ , the integral vanishes on the contour. It is easy to verify that $\nabla \times \partial\mathbf{V}_n/\partial\varphi = \partial(\nabla \times \mathbf{V}_n)/\partial\varphi$. Integrating over φ in the right-hand side of the preceding equation, we conclude that the second integral in (2.1) is equal to zero. The remaining terms in (2.1) expressed in terms of projections of the velocity on the directions ρ and φ can be written as

$$R \int\limits_{\Omega} \left\{ v_0 \left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2 + \left(\frac{\partial v_n}{\partial\varphi}\right)^2 \right] - 2 v_0 v_n \frac{\partial u_n}{\partial\varphi} + u_n \frac{\partial v_n}{\partial\varphi} \frac{d\rho v_0}{d\rho} \right\} d\rho\, d\varphi = 0$$

where u_n and v_n are components of the vector $\mathbf{V}_n$.

Integrating by parts and carrying out some simple manipulations, we can rewrite the latter equation as

$$R \int\limits_{\Omega} \left\{ v_0 \left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2 + \left(\frac{\partial v_n}{\partial\varphi}\right)^2 \right] + \frac{1}{2} \frac{d}{d\rho} \left[\frac{2v_0}{\rho} + \frac{1}{\rho} \frac{d\rho v_0}{d\rho} \right] (\rho u_n)^2 \right\} d\rho\, d\varphi = 0 \quad (2.2)$$

In the case where v_0 is given by Formula (1.5), Equation (2.2) can be rewritten as

$$\int\limits_{\Omega} \left\{ \frac{\rho^2 - 1}{\rho} \left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2 + \left(\frac{\partial v_n}{\partial\varphi}\right)^2 \right] + \frac{2}{\rho} u_n^2 \right\} d\rho\, d\varphi = 0$$

From the latter equation it is clear that $u_n \equiv 0$ and that v_n is independent of φ . In this case the projection on the direction φ of the first equation of boundary value problem (1.8) for $\mathbf{f}_n \equiv 0$ can be written as

$$\frac{1}{\rho} \frac{\partial p_n}{\partial\varphi} = \frac{d^2 v_n}{d\rho^2} + \frac{1}{\rho} \frac{dv_n}{d\rho} - \frac{v_n}{\rho^2} \quad (2.3)$$

Let us now make use of the condition of periodicity of the hydrodynamic pressure. The right-hand side of Equation (2.3) is independent of φ . Integrating (2.3) over $\varphi (0 \le \varphi < 2\pi)$ and making use of the periodicity of p_n with respect to φ , we obtain Equation

$$\frac{d^2 v_n}{d\rho^2} + \frac{1}{\rho} \frac{dv_n}{d\rho} - \frac{v_n}{\rho^2} = 0$$

This equation and the boundary conditions for v_n imply that $v_n \equiv 0$. Hence, in the case where the outer cylinder is the one which rotates, Equations (1.8) have a unique solution.

Now let us consider the case where the inner cylinder rotates. In this case v_0 is given by Formula (1.4) and Equation (2.2) can be rewritten as follows:

$$\int\limits_{\Omega} \frac{\beta^2 - \rho^2}{\rho} \left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2 + \left(\frac{\partial v_n}{\partial\varphi}\right)^2 \right] d\rho\, d\varphi = 2\beta^2 \int\limits_{\Omega} \frac{u_n^2}{\rho} d\rho\, d\varphi$$

In the latter equality we replace $\partial v_n/\partial\varphi$ in accordance with the continuity equation by the expression $-(u_n+\rho\partial u_n/\partial\rho)$, and obtain

$$\int\limits_\Omega\left\{\frac{\beta^2-\rho^2}{\rho^2}\left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2+\rho^2\left(\frac{\partial u_n}{\partial\rho}\right)^2+u_n{}^2\right]+(\beta^2-\rho^2)\frac{\partial u_n{}^2}{\partial\rho}\right\}d\rho\,d\varphi=2\beta^2\int\limits_\Omega\frac{u_n{}^2}{\rho}\,d\rho\,d\varphi$$

Hence, integrating by parts the term $(\beta^2-\rho^2)\partial u_n{}^2/\partial\rho$, we obtain

$$\int\limits_\Omega\left\{\frac{\beta^2-\rho^2}{\rho}\left[\left(\frac{\partial u_n}{\partial\varphi}\right)^2+\rho^2\left(\frac{\partial u_n}{\partial\rho}\right)^2\right]+\rho u_n{}^2\right\}d\rho\,d\varphi=\beta^2\int\limits_\Omega\frac{u_n{}^2}{\rho}\,d\rho\,d\varphi\leqslant\beta^2\int\limits_\Omega u_n{}^2\,d\rho\,d\varphi \qquad (2.4)$$

Let the function u_n satisfy inequality (2.4). It then satisfies the coarser inequality

$$\int\limits_\Omega\left[\frac{\beta^2-\rho^2}{\rho}\left(\frac{\partial u_n}{\partial\varphi}\right)^2+\rho u_n{}^2+(\beta+1)(\beta-\rho)\left(\frac{\partial u_n}{\partial\rho}\right)^2\right]d\rho\,d\varphi\leqslant\beta^2\int\limits_\Omega u_n{}^2\,d\rho\,d\varphi$$

and even more certainly the inequality

$$\int\limits_\Omega\left\{\frac{\beta^2-\rho^2}{\rho}\left(\frac{\partial u_n}{\partial\varphi}\right)^2+\rho u_n{}^2+[\beta+1-\beta^2(\beta-1)](\beta-\rho)\left(\frac{\partial u_n}{\partial\rho}\right)^2\right\}d\rho\,d\varphi\leqslant 0 \qquad (2.5)$$

obtained from the latter by replacing the right-hand side in accordance with the inequality

$$\int\limits_\Omega u_n{}^2\,d\rho\,d\varphi\leqslant(\beta-1)\int\limits_\Omega(\beta-\rho)\left(\frac{\partial u_n}{\partial\rho}\right)^2d\rho\,d\varphi$$

This inequality is valid for functions which vanish on the ring boundaries. For $\beta+1-\beta^2(\beta-1)\geq 0$ it follows from (2.5) that $u_n\equiv 0$, which implies, as above, that $v_n\equiv 0$. Thus, the solution of problem (1.8) is unique in this case as well. In the second case, uniqueness was successfully demonstrated for β satidfying the inequality $\beta+1-\beta^2(\beta-1)\geq 0$, i.e. for $\beta\in(1, 1.84)$. The problem under consideration has its principal application in the theory of high-speed support bearings, in which the indicated limitation as regards β is always fulfilled.

Let us rewrite problem (1.8) in operator form

$$A\,\mathbf{V}_n,\ H+B\,\mathbf{V}_n,\ H=\mathbf{f}_n \qquad (2.6)$$

The boundary value problem

$$\nabla\times(\nabla\times\mathbf{V}_n)+\nabla H_n=\mathbf{f}_n,\quad \nabla\mathbf{V}_n=0,\quad \mathbf{V}_n|_\Gamma=0$$

corresponds to the operator A .

This problem is investigated in [3], where it is shown that the operator A as an operator from $W_2{}^2$ in L_2 has the bounded inverse A^{-1}. The operator B is defined by (1.8); its domain of definition is broader than that of the operator A .

Multiplying (2.6) on the left by A^{-1} we arrive at Equation

$$\mathbf{V}_n,H+T\,\mathbf{V}_n,H=\mathbf{f}_n' \qquad (2.7)$$

where $\mathbf{f}_n'=A^{-1}\mathbf{f}_n,\ T=A^{-1}B$ is a completely continuous operator from $W_2{}^{\circ 2}$ in L_2 . This follows from the fact that A^{-1} is bounded, while B is completely continuous [4] as an operator from $W_2{}^{\circ 2}$ in L_2. The proven uniqueness of boundary value problem (1.8) implies, by virtue of Fredholm's theorems, its solvability and the estimate

$$\|\mathbf{V}_n\|_{W_2{}^2}\leqslant c\,\|\mathbf{f}_n\|_{L_2} \qquad (2.8)$$

Making use of the properties of a norm and the Cauchy inequality, we obtain from (1.9) the estimate

$$\|\mathbf{f}_n\|_{L_2}\leqslant C_1\left[\|\mathbf{V}_{n-1}\|_{W_2{}^2}+\sum_{k=1}^{n-1}\|\mathbf{V}_{n-k}\|_{L_4}\|\nabla\times\mathbf{V}_k\|_{L_4}+\|\boldsymbol{\chi}_n\|_{L_2}\right] \qquad (2.9)$$

Here c_1 is defined in terms of the functions Φ_2, Φ_3 and $\mathbf{V}_0$. From the imbedding theorems [4] and from (2.9) we have the estimate

$$\|\mathbf{f}_n\|_{L_2} \leqslant C_1 \left[\|\mathbf{V}_{n-1}\|_{W_2^2} + C_2 \sum_{k=1}^{n-1} \|\mathbf{V}_{n-k}\|_{W_2^2} \|\mathbf{V}_k\|_{W_2^2} + \|\chi_n\|_{L_2}\right] \tag{2.10}$$

where c_2 is a constant from the corresponding imbedding theorems.

Making use of (2.10), we can write (2.8) as

$$\|\mathbf{V}_n\|_{W_2^2} \leqslant C_3 \left[\|\mathbf{V}_{n-1}\|_{W_2^2} + \sum_{k=1}^{n-1} \|\mathbf{V}_{n-k}\|_{W_2^2} \|\mathbf{V}_k\|_{W_2^2} + \|\chi_n\|_{L_2}\right] \tag{2.11}$$

$$(C_3 = C \max (C_1, C_2C_1, 1))$$

3. Let us prove the convergence of series (1.7) relative to the parameter ϵ. From (1.7) we have

$$\|\mathbf{V}\|_{W_2^2} \leqslant \sum_{n=1}^{\infty} \|\mathbf{V}_n\|_{W_2^2} \varepsilon^n \tag{3.1}$$

where $\|\mathbf{V}_n\|_{W_2^2}$ are related to c $\|\mathbf{V}_k\|_{W_2^2} (k < n)$ by Expression (2.11).

Let us consider the algebraic equation

$$C_3 x^2 - x(1 - \varepsilon C_3) + C_3 \varepsilon \|\chi_1\|_{L_2} = 0 \tag{3.2}$$

For $\epsilon = 0$ this equation has as one of its roots $x = 0$. We shall attempt to find the solution of Equation (3.2) which is close to zero for small ϵ in the form

$$x = \sum_{n=1}^{\infty} x_n \varepsilon^n \tag{3.3}$$

Substituting (3.3) into (3.2) and collecting terms with like powers of the parameter ϵ, we obtain the recurrent relation

$$x_n = C_3 \left[x_{n-1} + \sum_{k=1}^{n-1} x_{n-k} x_k + \|\chi_n\|_{L_2}\right] \tag{3.4}$$

For $n = 1$, (2.11) and (3.4) imply the inequality $\|\mathbf{V}_1\|_{W_2^2} \leqslant x_1$.

Let the inequality $\|\mathbf{V}_m\|_{W_2^2} \leqslant x_m$ be valid for all $m \leq n - 1$. The inequality

$$\|\mathbf{V}_n\|_{W_2^2} \leqslant C_3 \left[\|\mathbf{V}_{n-1}\|_{W_2^2} + \sum_{k=1}^{n-1} \|\mathbf{V}_{n-k}\|_{W_2^2} \|\mathbf{V}_k\|_{W_2^2}\right] \leqslant C_3 \left[x_{n-1} + \sum_{k=1}^{n-1} x_{n-k} x_k\right] = x_n$$

is then valid for $\|\mathbf{V}_n\|_{W_2^2}$

This proves that series (3.3) majorizes series (3.1). The radius of convergence of series (3.3) can be easily determined from (3.2). Series (3.3) converges for ϵ which satisfy the inequality $2C_3\varepsilon |1 + 2C_3 \|\chi_1\|_{L_2} - 2\varepsilon| < 1$. Hence, series (3.1) converges under the same conditions.

BIBLIOGRAPHY

1. Kochin, N.E., Kibel', I.A. and Roze, N.V., Teoreticheskaia gidromekhanika (Theoretical Hydromechanics). Fizmatgiz, Moscow, Part 2, 1963.
2. Siryk, G.V., O konformnom otobrazhenii blizkikh oblastei (On the conformal mapping of close domains). Usp.mat.Nauk, Vol.11, № 5(71), 1965.
3. Ladyzhenskaia, O.A., Matematicheskie voprosy dinamiki viazkoi neszhimaemoi zhidkosti (Mathematical Problems of Viscous Incompressible Fluid Dynamics). Fizmatgiz, Moscow, 1961.
4. Smirnov, V.I., Kurs vysshei matematiki (Course in Higher Mathematics). Fizmatgiz, Moscow, Vol.5, 1959.

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